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A B S T R A C T

It is shown how to derive fractional supersymmetric quantum mechanics of order k as a superposition of $k-1$ copies of ordinary supersymmetric quantum mechanics.

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FRACTIONAL SUPERSYMMETRY AS A SUPERPOSITION OF ORDINARY SUPERSYMMETRY

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1 Introduction

In recent years, fractional supersymmetry has been the subject of numerous works. Indeed, k-fractional supersymmetry is closely connected to the notion of quantum algebra (deformation theory) and to the concept of intermediate statistics (of anyons [1] and k-fermions [2, 3]) interpolating between Bose-Einstein statistics and Fermi-Dirac statistics. Therefore, fractional supersymmetry constitutes a useful tool for dealing with anyonic statistics.

Fractional supersymmetric quantum mechanics of order k can be considered as an extension of ordinary supersymmetric quantum mechanics which corresponds to $k = 2$. An ordinary supersymmetric quantum-mechanical system may be generated from a doublet $(H, Q)_2$ of operators satisfying [4, 5]

$$Q^2 = 0,$$

$$QQ^\dagger + Q^\dagger Q = H.$$

The self-adjoint operator H and the operator Q act on a separable Hilbert space. The operator H is referred to as the Hamiltonian and the operator Q as the supersymmetry operator of the ordinary supersymmetric quantum-mechanical system. The operator Q gives rise to $\mathcal{N} = 2$ dependent supercharges $Q_- = Q$ and $Q_+ = Q^\dagger$ connected via Hermitean conjugation. They are nilpotent operators of order $k = 2$ and commute with the Hamiltonian H.

The *ordinary* supersymmetric quantum-mechanical system $(H, Q)_2$ can be extended to a *fractional* supersymmetric quantum-mechanical system $(H, Q)_k$ with $k \in \mathbf{N} \setminus \{0, 1, 2\}$

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as follows. The system $(H,Q)_k$ may be defined by [6, 7]

$$Q_- = Q, \quad Q_+ = Q^\dagger \quad (\Rightarrow\ Q_+ = Q_-^\dagger), \quad Q_\pm^k = 0, \tag{1a}$$

$$Q_-^{k-1}Q_+ + Q_-^{k-2}Q_+Q_- + \cdots + Q_+Q_-^{k-1} = Q_-^{k-2}H, \tag{1b}$$

$$[H, Q_\pm] = 0, \quad H = H^\dagger, \tag{1c}$$

where the self-adjoint operator H, the Hamiltonian of the system, and the $\mathcal{N} = 2$ supercharges Q_- and Q_+ act on a separable Hilbert space. Of course, the case $k = 2$ corresponds to an ordinary supersymmetric quantum-mechanical system.

In the present work, we study how it is possible to connect ordinary and k-fractional supersymmetric quantum-mechanical systems.

2 The algebra W_k

As an interesting question, we may ask: How to construct a fractional supersymmetric quantum-mechanical system of order k and, thus, fractional supersymmetric quantum mechanics of order k? This question can be answered through the definition of a generalized Weyl-Heisenberg algebra W_k. We now define the generic algebra W_k and shall see in the next section how a fractional supersymmetric quantum-mechanical system of order k may be associated with a given algebra W_k.

For k given, with $k \in \mathbf{N} \setminus \{0, 1\}$, the algebra W_k is generated by four linear operators X_-, X_+, N and K. The operators X_- and $X_+ = X_-^\dagger$ are shift operators connected via Hermitean conjugation. The operator N, called number operator, is self-adjoint. Finally, the operator K is a Z_k-grading unitary operator. The generators X_-, X_+, N and K satisfy [8]

$$[X_-, X_+] = \sum_{s=0}^{k-1} f_s(N)\, \Pi_s,$$

$$[N, X_-] = -X_-, \quad (+\text{h.c.}),$$

$$[K, X_-]_q = 0, \quad (+\text{h.c.}),$$

$$[K, N] = 0, \quad K^k = 1.$$

The functions $f_s : N \mapsto f_s(N)$ are such that $f_s(N)^\dagger = f_s(N)$, $[A, B]_q$ stands for $AB - qBA$, and the operators Π_s are defined by

$$\Pi_s = \frac{1}{k} \sum_{t=0}^{k-1} q^{-st}\, K^t$$

where

$$q = \exp\left(\frac{2\pi \mathrm{i}}{k}\right)$$

is a root of unity. To a given set $\{f_s : s = 0, 1, \cdots, k-1\}$ corresponds one algebra W_k.

The generalized Weyl-Heisenberg algebra W_k covers numerous algebras describing exactly solvable one-dimensional systems. The particular system corresponding to a given set $\{f_s : s = 0, 1, \cdots, k-1\}$ yields, in a Schrödinger picture, a particular dynamical system with a specific potential. Let us mention two interesting cases. The case

$$\forall s \in \{0, 1, \cdots, k-1\} \; : \; f_s(N) = f_s \text{ independent of } N$$

corresponds to systems with cyclic shape-invariant potentials (in the sense of Ref. [9]) and the case

$$\forall s \in \{0, 1, \cdots, k-1\} \; : \; f_s(N) = aN + b, \; (a, b) \in \mathbf{R}^2$$

to systems with translational shape-invariant potentials (in the sense of Ref. [10]). For instance, the case $(a = 0, b > 0)$ corresponds to the harmonic oscillator potential, the case $(a < 0, b > 0)$ to the Morse potential and the case $(a > 0, b > 0)$ to the Pöschl-Teller potential. For these various potentials, the part of W_k spanned by X_-, X_+ and N can be identified with the ordinary Weyl-Heisenberg algebra for $(a = 0, b \neq 0)$, with the su(2) Lie algebra for $(a < 0, b > 0)$ and with the su(1,1) Lie algebra for $(a > 0, b > 0)$.

3 A k-frational system associated with W_k

In order to associate a k-fractional supersymmetric quantum-mechanical system associated with a given generalized Weyl-Heisenberg algebra W_k, we must define a supersymmetry operator Q and an Hamiltonian H. The supersymmetry operator Q is defined by

$$Q \equiv Q_- = X_-(1 - \Pi_1) \Leftrightarrow Q^\dagger \equiv Q_+ = X_+(1 - \Pi_0).$$

Then, the Hamiltonian H associated with W_k can be deduced from Eq. (1b). This yields

$$H = (k-1)X_+X_- - \sum_{s=3}^{k} \sum_{t=2}^{s-1} (t-1) \, f_t(N - s + t) \, \Pi_s$$

$$- \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t-k) \, f_t(N - s + t) \, \Pi_s.$$

(Note that the summation from $s = k-2$ to $s = k$ appearing in some previous works by the authors [8] should be replaced by a summation from $s = 3$ to $s = k$.) It can be checked that H is self-adjoint and commutes with Q_- and Q_+. As a conclusion, the doublet $(H, Q)_k$ associated to W_k satisfies Eq. (1) and thus defines a k-fractional supersymmetric quantum-mechanical system.

4 Connection between fractional supersymmetry and ordinary supersymmetry

In order to establish a connection between *fractional* supersymmetric quantum mechanics of order k and *ordinary* supersymmetric quantum mechanics (corresponding to $k = 2$), it is necessary to construct sub-systems from the doublet $(H, Q)_k$ that correspond to ordinary supersymmetric quantum-mechanical systems. This may be achieved in the following way [11]. The general Hamiltonian H can be rewritten as

$$H = \sum_{s=1}^{k} H_s \, \Pi_s$$

where

$$H_s \equiv H_s(N) = (k-1)X_+X_- - \sum_{t=2}^{k-1}(t-1)\, f_t(N-s+t)$$
$$+(k-1)\sum_{t=s}^{k-1} f_t(N-s+t).$$

It can be shown that the operators $H_k \equiv H_0, H_{k-1}, \cdots, H_1$ turn out to be isospectral operators. It is possible to factorize H_s as [11]

$$H_s = X(s)_+ \, X(s)_-.$$

Let us now define: (i) the two (supercharge) operators

$$q(s)_- = X(s)_- \, \Pi_s, \quad q(s)_+ = X(s)_+ \, \Pi_{s-1}$$

and (ii) the (Hamiltonian) operator

$$h(s) = X(s)_- \, X(s)_+ \, \Pi_{s-1} + X(s)_+ \, X(s)_- \, \Pi_s.$$

It is then a simple matter of calculation to prove that $h(s)$ is self-adjoint and that

$$q(s)_+ = q(s)_-^\dagger, \quad q(s)_\pm^2 = 0, \quad h(s) = \{q(s)_-, q(s)_+\}, \quad [h(s), q(s)_\pm] = 0.$$

Consequently, the doublet $(h(s), q(s))_2$, with $q(s) \equiv q(s)_-$, satisfies Eq. (1) with $k = 2$ and thus defines an ordinary supersymmetric quantum-mechanical system (corresponding to $k = 2$).

The Hamiltonian $h(s)$ is amenable to a form more appropriate for discussing the link between ordinary supersymmetry and fractional supersymmetry. Indeed, we can show that

$$X(s)_- \, X(s)_+ = H_s(N+1).$$

Then, we can obtain the important relation

$$h(s) = H_{s-1} \, \Pi_{s-1} + H_s \, \Pi_s$$

to be compared with the expansion of H in terms of supersymmetric partners H_s.

As a result, the system $(H, Q)_k$, corresponding to k-fractional supersymmetry, can be described in terms of $k-1$ sub-systems $(h(s), q(s))_2$, corresponding to ordinary supersymmetry. The Hamiltonian $h(s)$ is given as a sum involving the supersymmetric partners H_{s-1} and H_s. Since the supercharges $q(s)_\pm$ commute with the Hamiltonian $h(s)$, it follows that

$$H_{s-1}X(s)_- = X(s)_-H_s, \quad H_sX(s)_+ = X(s)_+H_{s-1}.$$

As a consequence, the operators $X(s)_+$ and $X(s)_-$ render possible to pass from the spectrum of H_{s-1} and H_s to the one of H_s and H_{s-1}, respectively. This result is quite familiar for ordinary supersymmetric quantum mechanics (corresponding to $s = 2$).

For $k = 2$, the operator $h(1)$ is nothing but the total Hamiltonian H corresponding to ordinary supersymmetric quantum mechanics. For arbitrary k, the other operators $h(s)$ are simple replicas (except for the ground state of $h(s)$) of $h(1)$. In this sense, fractional supersymmetric quantum mechanics of order k can be considered as a set of $k-1$ replicas of ordinary supersymmetric quantum mechanics corresponding to $k = 2$ and typically described by $(h(s), q(s))_2$. As a further argument, it is to be emphasized that

$$H = q(2)_- \, q(2)_+ + \sum_{s=2}^{k} q(s)_+ \, q(s)_-$$

which can be identified with $h(2)$ for $k = 2$.

5 Conclusions

Starting from a Z_k-graded algebra W_k, charaacterized by a set $\{f_s : s = 0, 1, \cdots, k-1\}$ of structure functions, it was shown how to associate a k-fractional supersymmetric quantum-mechanical system $(H, Q)_k$ characterized by an Hamiltonian H and a supercharge Q.

The Hamiltonian H for the system $(H, Q)_k$ was developed as a superposition of k isospectral supersymmetric partners $H_k, H_{k-1}, \cdots, H_1$. It was proved that the system $(H, Q)_k$ can be described in terms of $k-1$ sub-systems $(h(s), q(s))_2$ which are ordinary supersymmetric quantum-mechanical systems.

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